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On the Dirichlet and the Mixed Dirichlet-Neumann Problems in Exterior Domains

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ABSTRACT

We study the properties of generalized solutions in unbounded domains and the asymptotic behavior of solutions of elliptic boundary value problems at infinity. Moreover, we study the unique solvability of the Dirichlet and the mixed Dirichlet-Neumann biharmonic problems in the exterior of a compact set under the assumption that generalized solutions of these problems has a bounded Dirichlet (energy) integral with weight |x|^a. We used the variation principle and depending on the value of the parameter a, we obtained uniqueness (non-uniqueness) theorems of these problems
or present exact formulas for the dimension of the space of solutions.

Keywords: *Biharmonic operator, mixed Dirichlet-Neumann problem, weighted Dirichlet integral, Sobolev spaces.*
MSC: *35J35; 35J40; 31B30*

1. INTRODUCTION

Let Ω be an unbounded domain in $\mathbb{R}^n$, $n \geq 2$, $\Omega = \mathbb{R}^n \setminus G$ with the boundary $\partial\Omega \in C^2$, where G is a bounded simply connected domain (or a union of finitely many such domains) in $\mathbb{R}^n$, $0 \in G$, $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

In the domain Ω we consider the following mixed problems for the biharmonic equation

$$\Delta^2 u = 0 \tag{1}$$

with the Dirichlet boundary conditions

$$u\big|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{2}$$

and the Dirichlet–Neumann boundary conditions

$$u\big|_{\Gamma_1} = \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} = 0, \quad \Delta u\big|_{\Gamma_2} = \frac{\partial \Delta u}{\partial \nu}\Big|_{\Gamma_2} = 0, \tag{3}$$

where $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\mathrm{mes}_{n-1}\Gamma_1 \neq 0$, $\nu = (\nu_1, \dots, \nu_n)$ is the outer unit normal vector to $\partial\Omega$.

As is well known, if Ω is an unbounded domain, one should additionally characterize the behavior of the solution at infinity. As a rule, to this end, one usually poses either the condition that the Dirichlet (energy) integral is finite or a condition on the character of vanishing of the modulus of the solution as $|x| \to \infty$. Such conditions at infinity are natural and were studied by several authors (e.g., [10]– [12]).

The behavior of solutions of the Dirichlet problem for the biharmonic equation as $|x| \to \infty$ was considered in [7], [8], where estimates for $|u(x)|$ and $|\nabla u(x)|$ as $|x| \to \infty$ were obtained under certain geometric conditions on the domain boundary.

Elliptic problems with parameters in the boundary conditions have been called Steklov or Steklov-type problems since their first appearance in [30]. For the biharmonic operator, these conditions were first considered in [1], [13] and [28], whose authors the isoperimetric properties of the first eigenvalue were studied.

Note that standard elliptic regularity results are available in [4]. The monograph covers higher order linear and nonlinear elliptic boundary value problems, mainly with the biharmonic or polyharmonic operator as leading principal part. The underlying models and, in particular, the role of different boundary conditions are explained in detail. As for linear problems, after a brief summary of the existence theory and L^p and Schauder estimates, the focus is on positivity. The required kernel estimates are also presented in detail.

In [3] and [4], the spectral and positivity preserving properties for the inverse of the biharmonic operator under Steklov and Navier boundary conditions are studied. These are connected with the first Steklov eigenvalue. It is shown that the positivity preserving property is quite sensitive to the parameter involved in the boundary condition. Moreover, positivity of the Steklov boundary value problem is linked with positivity under boundary conditions of Dirichlet and Navier type.

In [2], the boundary value problems for the biharmonic equation and the Stokes system are studied in a half space, and, using the Schwartz reflection principle in weighted L^q -space, the uniqueness of solutions of the Stokes system or the biharmonic equation is proved.

In the present note, this condition is the boundedness of the weighted Dirichlet integral:

$$D_a(u,\Omega) \equiv \int_\Omega |x|^a \sum_{|\alpha|=2} |\partial^\alpha u|^2\, dx < \infty, \quad a \in \mathbb{R}.$$

In various classes of unbounded domains with finite weighted Dirichlet (energy) integral, one of the author [14]– [26] studied uniqueness (non–uniqueness) problem and found the dimensions of the spaces of solutions of boundary value problems for the elasticity system and the biharmonic (polyharmonic) equation.

By developing an approach based on the use of Hardy type inequalities [6], [10]– [12], in the present note, we obtain a uniqueness (non–uniqueness) criterion for a solution of the Dirichlet and the mixed Dirichlet–Neumann problems for the biharmonic equation.

Notation: $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions in Ω with compact support in Ω.

We denote by $H^m(\Omega,\Gamma)$, $\Gamma \subset \overline{\Omega}$, the Sobolev space of functions in Ω obtained by the completion of $C^\infty(\overline{\Omega})$ vanishing in a neighborhood of Γ with respect to the norm

$$||u; H^m(\Omega,\Gamma)|| = \left(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha u|^2 dx\right)^{1/2}, \quad m = 1, 2,$$

where $\partial^\alpha \equiv \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\alpha_i \ge 0$ are integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$; if $\Gamma = \emptyset$, we denote $H^m(\Omega,\Gamma)$ by $H^m(\Omega)$.

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$\overset{\circ}{H}{}^{m}(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u(x); H^m(\Omega)\|$;

$\overset{\circ}{H}{}^{m}_{loc}(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the family of semi-norms

$$\|u; H^m(\Omega \cap B_0(R))\| = \left(\int\limits_{\Omega \cap B_0(R)} \sum_{|\alpha| \le m} |\partial^\alpha u|^2 \, dx \right)^{1/2}$$

for all open balls $B_0(R) := \{x : |x| < R\}$ in $\mathbb{R}^n$ for which $\Omega \cap B_0(R) \neq \emptyset$.

Let $\binom{n}{k}$ be the (n,k) - binomial coefficient, $\binom{n}{k}$=0 for $k > n$.

2. Definitions and auxiliary statements

Definition 2.1. *A solution of the homogenous biharmonic equation (1) in Ω is a function $u \in H^2_{loc}(\Omega)$ such that, for every function $\varphi \in C_0^\infty(\Omega)$, the following integral identity holds:*

$$\int_\Omega \Delta u \, \Delta \varphi \, dx = 0.$$

Lemma 2.2. *Let u be a solution of equation (1) in Ω such that $D_a(u, \Omega) < \infty$. Then*

$$u(x) = P(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x), \quad x \in \Omega, \tag{4}$$

where $P(x)$ is a polynomial, $\operatorname{ord} P(x) < m_0 = \max\{2, 2 - n/2 - a/2\}$, $\beta_0 = 2 - n/2 + a/2$, $\Gamma(x)$ is the fundamental solution of equation (1), $C_\alpha = \text{const}$, $\beta \ge 0$ is an integer, and the function u^β satisfies the estimate:

$$|\partial^\gamma u^\beta(x)| \le C_{\gamma\beta} |x|^{3-n-\beta-|\gamma|}, \, C_{\gamma\beta} = \text{const},$$

for every multi-index γ.

Remark 2.3. *As is known [29], the fundamental solution $\Gamma(x)$ of the biharmonic equation has the form*

$$\Gamma(x) = \begin{cases} C|x|^{4-n}, \; if\, 4 - n < 0 \, or \, n \, is \, odd, \\ C|x|^{4-n} \ln |x|, \; if \, 4 - n \ge 0 \, and \, n \, is \, even. \end{cases}$$

Proof of Lemma 2.2. Consider the function $v(x) = \theta_N(x) u(x)$, where $\theta_N(x) = \theta(|x|/N), \theta \in C^\infty(\mathbb{R}^n)$, $0 \le \theta \le 1$, $\theta(s) = 0$ for $s \le 1$, $\theta(s) = 1$ for $s \ge 2$, while $N \gg 1$ and $G \subset \{x : |x| < N\}$. We extend v to $\mathbb{R}^n$ by setting $v = 0$ on $G = \mathbb{R}^n \setminus \overline{\Omega}$.

Then the function v belongs to $C^\infty(\mathbb{R}^n)$ and satisfies the equation

$$\Delta^2 v = f,$$

where $f \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp} f \subset \{x : |x| < 2N\}$. It is easy to see that $D_a(v, \mathbb{R}^n) < \infty$.

We can now use Theorem 1 of [9] since it is based on Lemma 2 of [9], which imposes no constraint on the *sign* of σ. Hence, the expansion

$$v(x) = P(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + v^\beta(x),$$

holds for each a, where $P(x)$ is a polynomial of order $\operatorname{ord} P(x) < m_0 = \max\{2, 2 - n/2 - a/2\}$, $\beta_0 = 2 - n/2 + a/2$, $C_\alpha = \text{const}$ and

$$|\partial^\gamma v^\beta(x)| \le C_{\gamma\beta} |x|^{3-n-\beta-|\gamma|}, \quad C_{\gamma\beta} = \text{const}.$$

Therefore, by the definition of v, we obtain (4). The proof of Lemma 2.2 is complete.

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Definition 2.4. *A function u is a solution of the Dirichlet problem (1), (2), if $u \in H^2_{loc}(\Omega)$ such that for every function $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 0$ in the neighborhood of infinity, the following integral identity holds:*

$$\int_\Omega \Delta u \, \Delta \varphi \, dx. \tag{5}$$

Definition 2.5. *A function u is a solution of the mixed Dirichlet–Neumann problem (1), (3), if $u \in \overset{\circ}{H}{}^2_{loc}(\Omega, \Gamma_1)$ such that for every function $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 0$ in the neighborhood of Γ_1, the following integral identity (5) holds.*

3. Main Results

Theorem 3.1. *The Dirichlet problem (1),(2) with the condition $D(u, \Omega) < \infty$ has $n+1$ linearly independent solutions.*

Proof. For any nonzero vector A in $\mathbb{R}^n$, we construct a generalized solution u_A of the biharmonic equation (1) with the boundary conditions

$$u_A(x)\big|_{\partial\Omega} = (Ax)\big|_{\partial\Omega}, \quad \frac{\partial u_A(x)}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial (Ax)}{\partial \nu}\Big|_{\partial\Omega}, \tag{6}$$

and the condition

$$\chi(u_A, \Omega) \equiv \begin{cases} \displaystyle\int_\Omega \left(\frac{|u_A(x)|^2}{|x|^4} + \frac{|\nabla u_A(x)|^2}{|x|^2} + |\nabla\nabla u_A(x)|^2 \right) dx < \infty \\ \hfill \text{for } n > 4, \\ \displaystyle\int_\Omega \left(\frac{|u_A(x)|^2}{||x|^2 \ln |x||^2} + \frac{|\nabla u_A(x)|^2}{||x| \ln |x||^2} + |\nabla\nabla u_A(x)|^2 \right) dx < \infty \\ \hfill \text{for } 2 \le n \le 4, \end{cases} \tag{7}$$

for $A, x \in \mathbb{R}^n$, where Ax denotes the standard scalar product of A and x.

Such a solution of problem (1), (6) can be constructed by the variational method [29], minimizing the functional

$$\Phi(v) = \frac{1}{2} \int_\Omega |\Delta v|^2 \, dx$$

in the class of admissible functions $\{v \colon v \in H^2(\Omega),\, v(x)\big|_{\partial\Omega} = (Ax)\big|_{\partial\Omega},\, \frac{\partial v(x)}{\partial \nu}\big|_{\partial\Omega} = \frac{\partial (Ax)}{\partial \nu}\big|_{\partial\Omega};\, v$ is compactly supported in $\overline{\Omega}\}$.

The validity of condition (7) as a consequence of the Hardy inequality follows from the results in [10]– [12].

Now, for any arbitrary number $e \neq 0$, we construct a generalized solution u_e of equation (1) with the boundary conditions

$$u_e\big|_{\partial\Omega} = e, \quad \frac{\partial u_e}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{8}$$

and the condition

$$\chi(u_e, \Omega) \equiv \begin{cases} \displaystyle\int_\Omega \left(\frac{|u_e(x)|^2}{|x|^4} + \frac{|\nabla u_e(x)|^2}{|x|^2} + |\nabla\nabla u_e(x)|^2 \right) dx < \infty \\ \hfill \text{for } n > 4, \\ \displaystyle\int_\Omega \left(\frac{|u_e(x)|^2}{||x|^2 \ln |x||^2} + \frac{|\nabla u_e(x)|^2}{||x| \ln |x||^2} + |\nabla\nabla u_e(x)|^2 \right) dx < \infty \\ \hfill \text{for } 2 \le n \le 4. \end{cases} \tag{9}$$

The solution of problem (1), (8) also is constructed by the variational method with the minimization of the corresponding functional in the class of admissible functions $\{v : v \in H^2(\Omega),\, v|_{\partial\Omega} = e,\ \frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0,\ \ v$ is compactly supported in $\overline{\Omega}\}$.

The condition (9) as a consequence of the Hardy inequality follows from the results in [10]–[12].

Consider the function $v(x) = (u_A(x) - Ax) - (u_e - e)$.

Obviously, v is a solution of problem (1), (2):

$$\Delta^2 v = 0, \quad x \in \Omega, \quad v\big|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = 0.$$

One can easily see that $v \not\equiv 0$ and $D(v, \Omega) < \infty$.

To each nonzero vector $\mathbf{A} = (A_0, A_1, \dots, A_n)$ in $\mathbb{R}^{n+1}$, there corresponds a nonzero solution $v_{\mathbf{A}} = (v_{A_0}, v_{A_1}, \dots, v_{A_n})$ of problem (1), (2) with the condition $D(v_{\mathbf{A}}, \Omega) < \infty$, and moreover,

$$v_{\mathbf{A}}(x) = u_A(x) - u_e - Ax + e.$$

Let $A_0, A_1, \dots, A_n$ be a basis in $\mathbb{R}^{n+1}$. Let us prove that the corresponding solutions $v_{A_0}, v_{A_1}, \dots, v_{A_n}$ are linearly independent. Let

$$\sum_{i=0}^{n} C_i v_{A_i} \equiv 0, \qquad C_i = \text{const}\,.$$

Set $W(x) \equiv \sum_{i=1}^{n} C_i A_i x - C_0 e$. We have

$$W(x) = \sum_{i=1}^{n} C_i u_{A_i}(x) - C_0 u_e,$$

$$\int_\Omega |x|^{-2} |\nabla W|^2 \, dx < \infty, \quad n > 4,$$

$$\int_\Omega ||x| \ln |x||^{-2} |\nabla W|^2 \, dx < \infty, \quad 2 \le n \le 4.$$

Let us show that

$$W(x) \equiv \sum_{i=1}^{n} C_i A_i x - C_0 e \equiv 0.$$

Let $T = \sum_{i=0}^{n} C_i A_i = (t_0, \dots, t_n)$, where $A_0 = -e$. Then

$$\int_\Omega |x|^{-2} |\nabla W|^2 \, dx = \int_\Omega |x|^{-2} (t_1^2 + \cdots + t_n^2) \, dx = \infty, \quad n > 4,$$

$$\int_\Omega ||x| \ln |x||^{-2} |\nabla W|^2 \, dx = \int_\Omega ||x| \ln |x||^{-2} (t_1^2 + \cdots + t_n^2) \, dx = \infty, \quad 2 \le n \le 4,$$

if $T \ne 0$.

Consequently, $T = \sum_{i=0}^{n} C_i A_i = 0$, and since the vectors $A_0, A_1, \dots, A_n$ are linearly independent, we obtain $C_i = 0,\ i = 0, 1, \dots, n$.

Thus, the Dirichlet problem (1), (2) with the condition $D(u, \Omega) < \infty$ has at least $n + 1$ linearly independent solutions.

Let us prove that each solution u of problem (1), (2) with the condition $D(u, \Omega) < \infty$ can be represented as a linear combination of the functions $v_{A_0}, v_{A_1}, \dots, v_{A_n}$, i.e.

$$u = \sum_{i=0}^{n} C_i v_{A_i}, \qquad C_i = \text{const}\,.$$

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Since $A_0, A_1, \ldots, A_n$ is a basis in $\mathbb{R}^{n+1}$, it follows that there exists constants $C_0, C_1, \ldots, C_n$ such that

$$A = \sum_{i=0}^{n} C_i A_i.$$

We set

$$u_0 \equiv u - \sum_{i=0}^{n} C_i v_{A_i}.$$

Obviously, the function u_0 is a solution of problem (1), (2), and $D(u_0, \Omega) < \infty$, $\chi(u_0, \Omega) < \infty$.

Let us show that $u_0 \equiv 0$, $x \in \Omega$. To this end, we substitute the function $\varphi(x) = u_0(x)\theta_N(x)$ into the integral identity (5) for the function u_0, where $\theta_N(x) = \theta(|x|/N)$, $\theta \in C^\infty(\mathbb{R})$, $0 \le \theta \le 1$, $\theta(s) = 0$ for $s \ge 2$ and $\theta(s) = 1$ for $s \le 1$; then we obtain

$$\int_\Omega (\Delta u_0)^2 \theta_N(x)\, dx = -J_1(u_0) - J_2(u_0), \tag{10}$$

where

$$J_1(u_0) = 2 \int_\Omega \Delta u_0 \, \nabla u_0 \, \nabla \theta_N(x)\, dx, \qquad J_2(u_0) = \int_\Omega u_0 \, \Delta u_0 \, \Delta \theta_N(x)\, dx.$$

By applying the Cauchy–Schwarz inequality and by taking into account the conditions $D(u_0, \Omega) < \infty$ and $\chi(u_0, \Omega) < \infty$, one can easily show that $J_1(u_0) \to 0$ and $J_2(u_0) \to 0$ as $N \to \infty$. Consequently, by passing to the limit as $N \to \infty$ in (10), we obtain

$$\int_\Omega (\Delta u_0)^2\, dx = 0.$$

Therefore, we have

$$\Delta u_0 = 0, \quad x \in \Omega,$$

$$u_0\big|_{\partial\Omega} = \frac{\partial u_0}{\partial \nu}\bigg|_{\partial\Omega} = 0.$$

Hence, it follows [5, Ch.2] that $u_0 \equiv 0$ in Ω. The proof of the theorem is complete. □

Theorem 3.2. *The Dirichlet problem (1), (2) with the condition $D_a(u, \Omega) < \infty$ has:*

(i) the trivial solution for $n - 2 \le a < \infty$, $n > 4$;

(ii) n linearly independent solutions for $n - 4 \le a < n - 2$, $n > 4$;

(iii) $n + 1$ linearly independent solutions for $-n \le a < n - 4$, $n > 4$;

(iv) $k(r, n)$ linearly independent solutions for $-2r + 2 - n \le a < -2r + 4 - n$, $r > 1$, $n > 4$, where

$$k(r, n) = \binom{r+n}{n} - \binom{r+n-4}{n}.$$

Complete proof of Theorem 3.2 carried out in [15].

Theorem 3.3. *The mixed Dirichlet–Neumann problem (1),(3) with the condition $D(u, \Omega) < \infty$ has $n + 1$ linearly independent solutions.*

The proof of Theorem 3.3 is also carried out as in Theorem 3.1.

Theorem 3.4. *The mixed Dirichlet–Neumann problem (1), (3) with the condition $D_a(u,\Omega) < \infty$ has:*

(i) the trivial solution for $n-2 \le a < \infty$, $n > 4$;

(ii) n linearly independent solutions for $n-4 \le a < n-2$, $n > 4$;

(iii) $n+1$ linearly independent solutions for $-n \le a < n-4$, $n > 4$;

(iv) $k(r,n)$ linearly independent solutions for $-2r+2-n \le a < -2r+4-n$, $r > 1$, $n > 4$, where

$$k(r,n) = \binom{r+n}{n} - \binom{r+n-4}{n}.$$

The proof of Theorem 3.4 is based on Lemma 2.2 about the asymptotic expansion of the solution of the biharmonic equation and the Hardy type inequalities for unbounded domains [10]– [12]. In case (iv), we need to determine the number of linearly independent solutions of the biharmonic equation (1), the degree of which not exceed the fixed number.

It is well know that the dimension of the space of all polynomials in $\mathbb{R}^n$ of degree $\le r$ is equal $\binom{r+n}{n}$ [27]. Then the dimension of the space of all biharmonic polynomials in $\mathbb{R}^n$ of degree $\le r$ is equal to

$$\binom{r+n}{n} - \binom{r+n-4}{n},$$

since the biharmonic equation is the vanishing of some polynomial of degree $r-4$ in $\mathbb{R}^n$. If we denote by $k(r,n)$ the number of linearly independent polynomial solutions of equation (1) whose degree do not exceed r and by $l(r,n)$ the number of linearly independent homogeneous polynomials of degree r, that are solutions of equation (1), then

$$k(r,n) = \sum_{s=0}^{r} l(s,n),$$

where

$$l(s,n) = \binom{s+n-1}{n-1} - \binom{s+n-5}{n-1}, \qquad s > 0.$$

Further, we prove that the Dirichlet problem (1), (3) with the condition $D_a(u,\Omega) < \infty$ for $-2r+2-n \le a < -2r+4-n$ has equally $k(r,n)$ of linearly independent solutions.

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